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The generalized Bernstein problem on weighted Lacunary polynomial approximation☆

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Abstract

We obtain a necessary and sufficient condition for the lacunary polynomials to be dense in weighted L^p spaces of functions on the real line. This generalizes the solution to the classical Bernstein problem given by Izumi, Kawata and Hall.

1. Introduction

In this paper, we will study weighted lacunary polynomial approximation that generalizes the classical Bernstein problem [2]. First we introduce some notations for convenience of the readers before we state our main results.

Let a *weight* α be an even nonnegative function continuous on $\mathbb{R}$ such that

$$\lim_{|t|\to\infty} \frac{\alpha(t)}{\log|t|} = \infty. \tag{1}$$

Consider the Banach spaces

$$L^p_\alpha = \left\{ f : \|f\|_{p,\alpha} = \left(\int_{-\infty}^{+\infty} |f(t)e^{-\alpha(t)}|^p \, dt \right)^{\frac{1}{p}} < +\infty \right\}, 1 \leqslant p < +\infty$$

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$$C_\alpha = \left\{ f \in C(\mathbb{R}) : \lim_{|t| \to +\infty} |f(t)| e^{-\alpha(t)} = 0, \|f\|_\alpha = \sup_{t \in \mathbb{R}} \{|f(t) e^{-\alpha(t)}|\} \right\}.$$

The classical Bernstein problem is to find out whether the polynomials are dense in C_α. Several solutions are proposed by mathematicians [1,7,9,10,14]. For the surveys on it see [1,11,14] and for the recent progress see Borichev [4,5]. Here we are interested in the results obtained by Izumi and Kawata [10] and by Hall [9]. We can formulate their results as follows:

Theorem A. *Suppose that $\alpha(t)$ is an even nonnegative function satisfying* (1) *and $\alpha(e^t)$ is convex on $\mathbb{R}$. Then a necessary and sufficient condition for the polynomials to be dense in the space C_α is*

$$\int_{-\infty}^{+\infty} \frac{\alpha(t)}{1+t^2}\, dt = \infty. \tag{2}$$

We note that Koosis [11] pointed out that Theorem A holds with L^p_α instead of C_α and the proof is essentially the same.

Inspired by Müntz Theorem [6], it is natural to consider the density of the lacunary polynomials in the space L^p_α and C_α. Denote by $M(\Lambda)$ the space of the lacunary polynomials which are finite linear combinations of the system $\{t^\lambda : \lambda \in \Lambda\}$, where $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ is a sequence of increasing nonnegative integers. Our condition (1) guarantees that $M(\Lambda)$ is a subspace of L^p_α and C_α, we then ask whether $M(\Lambda)$ is dense in L^p_α and C_α in the respect norms—this is so-called the generalized Bernstein problem on weighted lacunary polynomial approximation.

Motivated by the classical Bernstein problem and Malliavin's method [13], we find a necessary and sufficient condition for the generalized Bernstein problem. We state our main conclusions as follows:

Theorem 1. *Suppose that $\alpha(t)$ is an even nonnegative function satisfying* (1) *and $\alpha(e^t)$ is a convex function on $\mathbb{R}$. Let $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ be a sequence of increasing nonnegative integers. Then $M(\Lambda)$ is dense in $L^p_\alpha (1 \leqslant p < +\infty)$ if and only if*

$$\int_1^{+\infty} \frac{\alpha(\exp\{\lambda_1(t) - a\})}{t^2}\, dt = +\infty \text{ and } \int_1^{+\infty} \frac{\alpha(\exp\{\lambda_2(t) - a\})}{t^2}\, dt = +\infty \tag{3}$$

for each $a \in \mathbb{R}$, where

$$\lambda_1(r) = 2 \sum_{\lambda_n \leqslant r, \lambda_n \ odd} \frac{1}{\lambda_n}, r \geqslant \lambda_1;\ \lambda_1(r) = 0, r < \lambda_1, \tag{4}$$

$$\lambda_2(r) = 2 \sum_{\lambda_n \leqslant r, \lambda_n \ even} \frac{1}{\lambda_n}, r \geqslant \lambda_1;\ \lambda_2(r) = 0, r < \lambda_1. \tag{5}$$

Remark 2. Theorem 1 holds for C_α with $\Lambda \bigcup \{0\}$ instead of Λ. If $\Lambda = \mathbb{N} = \{1, 2, \ldots\}$, then $\lambda_1(r) = \lambda_2(r) = \log r + O(1)(r \to \infty)$ and hence condition (3) is equivalent to (2). Therefore Theorem A follows from our theorem.

In one previous paper [8] , we conjectured that the condition

$$\int_1^{+\infty} \frac{\alpha(\exp\{k.(t)-a\})}{t^2}\, dt = \infty, \tag{6}$$

where $k.(r) = \inf\{k(r') : r' \geqslant r\}$, $k(r) = \lambda(r) - \log^+ r$, $\log^+ r = \max\{\log r, 0\}$ and $\lambda(r) = \lambda_1(r) + \lambda_2(r)$, is a necessary and sufficient condition. Thank Professor Alexander Borichev who points out that it is not necessary and then we give condition (3) here, by which one can easily see that condition (6) is sufficient but not necessary.

Remark 3. In [12], Kroó and Szabados considered the weighted approximation by general lacunary polynomials with real exponents in the space of functions continuous on the half-axis and they proved a sufficient condition which implies (3). In fact, we can even consider weighted approximation by more general polynomial system $\{t^\lambda : \lambda \in \mathbb{C}\}$ on the half-axis, but we can not do that on the whole real axis. We will discuss this problem in one forthcoming paper.

2. Proof of Theorem 1

In order to prove Theorem 1, we need the following two technical lemmas:

Lemma 4 (*Malliavin [13]*). *Let $\beta(t)$ be a nonnegative convex function on $\mathbb{R}$ satisfying* (1), *and assume that*

$$\beta^*(t) = \sup\{xt - \beta(x) : x \in \mathbb{R}\},\ t \in \mathbb{R} \tag{7}$$

is the Young transform [15] *of the function $\beta(x)$. Suppose that $\lambda(r)$ is an increasing function on* $[0, \infty)$ *satisfying*

$$\lambda(R) - \lambda(r) \leqslant A(\log\ R - \log\ r + 1)\ \ (R > r > 1). \tag{8}$$

Then there exists an analytic function $f(z) \not\equiv 0$ in $\mathbb{C}_+$ satisfying

$$|f(z)| \leqslant A \exp\{Ax + \beta(x) - x\lambda(|z|)\},\ z = x + iy \in \mathbb{C}_+, \tag{9}$$

if and only if there exists $a \in \mathbb{R}$ such that

$$\int_0^{+\infty} \frac{\beta^*(\lambda(t)-a)}{1+t^2}\, dt < \infty. \tag{10}$$

Remark 5. Lemma 1 is a result of Malliavin's uniqueness theorem about Watson's problem (hereafter we denote a positive constant by A, not necessarily the same at each occurrence).

Lemma 6 (*Boas [3]*). *If $\Lambda = \{\lambda_n\}_{n=1}^{+\infty}$ is a sequence of positive increasing integers, then the function*

$$G_\Lambda(z) = \prod_{n=1}^{\infty} \left(\frac{\lambda_n - z}{\lambda_n + z}\right) \exp\left(\frac{2z}{\lambda_n}\right) \tag{11}$$

is analytic in the closed right half plane $\overline{\mathbb{C}}_+ = \{z = x + iy : x \geqslant 0\}$, *and there exists a positive constant A such that*

$$|G_\Lambda(z)| \leqslant \exp\{x\lambda(|z|) + Ax\}, \quad z = x + iy \in \mathbb{C}_+, \tag{12}$$

$$|G_\Lambda(z)| \geqslant \exp\{x\lambda(|z|) - Ax\}, \quad z = x + iy \in \Sigma(\Lambda), \tag{13}$$

where $\Sigma(\Lambda) = \{z \in \mathbb{C}_+ : |z - \lambda_n| \geqslant 1/4\}$ *and* $\lambda(r)$ *is defined as follows*:

$$\lambda(r) = 2 \sum_{\lambda_n \leqslant r} \frac{1}{\lambda_n}, r \geqslant \lambda_1; \lambda(r) = 0, r < \lambda_1. \tag{14}$$

Remark 7. Lemma 5 is proved by Fuchs.

Now we will start to prove Theorem 1.

Sufficiency of Theorem 1. First we can identify bounded linear functionals on $L^p_\alpha, 1 \leqslant p < +\infty$, with elements of $L^q_{-\alpha}$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$L^q_{-\alpha} = \left\{ f : \|f\|_{q,-\alpha} = \left(\int_{-\infty}^{+\infty} |f(t)e^{\alpha(t)}|^q \, dt \right)^{\frac{1}{q}} < +\infty \right\}, 1 < q < +\infty,$$

$$L^\infty_{-\alpha} = \left\{ f : \|f\|_{\infty,-\alpha} = \operatorname{ess\,sup}\{|f(t)|e^{\alpha(t)} : x \in \mathbb{R}\} < +\infty \right\},$$

with the duality being

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)g(x)\, dx, \ f \in L^p_\alpha, g \in L^q_{-\alpha}.$$

Then by Hahn–Banach theorem, we only need to prove that

$$\langle t^\lambda, g \rangle = 0, \forall \lambda \in \Lambda \tag{15}$$

implies $g(t) = 0$ a.e. Fix $g \in L^q_{-\alpha}$ such that (15) holds. Put

$$g_1(t) = (g(t) + g(-t))/2, \ g_2(t) = (g(t) - g(-t))/2.$$

And we get that $g_1, g_2 \in L^p_{-\alpha}$ and the following relations hold:

$$g_1(-t) = g_1(t), g_2(-t) = -g_2(t), g(t) = g_1(t) + g_2(t), t \in \mathbb{R}.$$

Hence we turn to prove that $g_1(t) = g_2(t) = 0$ a.e. We associate with g_k the functions

$$f_k(z) = 2 \int_0^{+\infty} t^z g_k(t)\, dt, \operatorname{Re} z > 0 \ (k = 1, 2).$$

For $t > 0$, $t^z = \exp\{z \log t\}$, by definition. We claim that f_k is holomorphic in the right half plane. Furthermore, (15) and the definition of $g_k(z)$ show that

$$f_k(\lambda) = 0, \lambda \in \Lambda_k \ (k = 1, 2), \tag{16}$$

where $\Lambda_1 = \{\lambda \in \Lambda : \lambda \text{ is odd}\}$ and $\Lambda_2 = \{\lambda \in \Lambda : \lambda \text{ is even}\}$. The fact that $g_k(t) \in L^p_{-\alpha}$ suggests that

$$|f_k(z)| \leqslant A e^{\beta(x)}, z = x + iy \in \mathbb{C}_+, \tag{17}$$

where

$$\beta(x) = \sup\{x \log t - \alpha(t) : t > 0\} \tag{18}$$

is Young transform [15] of the convex function $\alpha(e^s)$. Define

$$F_k(z) = f_k(z)/G_k(z), z = x + iy \in \mathbb{C}_+, \tag{19}$$

where $G_k(z) = \prod_{\lambda \in \Lambda_k} \frac{\lambda - z}{\lambda + z} \exp(\frac{2z}{\lambda})$. Then by (13) and (17) and according to maximum modulus principle we have that

$$|F_k(z)| \leqslant A \exp\{\beta(x) - x\lambda_k(r) + Ax\}, \quad z \in \mathbb{C}_+ \ \ (k = 1, 2). \tag{20}$$

We may assume, without loss of generality, that $\alpha(1) = 0$. As is known [15], $\beta(x)$ is a convex nonnegative function which also satisfies $\beta(0) = 0$ and

$$\sup\{xs - \beta(x) : x \geqslant 0\} = \alpha(e^s). \tag{21}$$

Now by Lemma 1, condition (3) yields that $F_k(z) \equiv 0$ and hence $f_k(z) \equiv 0$ $(k = 1, 2)$. Therefore $g_1(t) = g_2(t) = 0$ a.e. and hence $g(t) = 0$ a.e. This completes the proof of sufficiency of the theorem.

Necessity of Theorem 1. Equivalently, we prove that:

If there exists $b \in \mathbb{R}$ such that

$$\int_1^{+\infty} \frac{\alpha(\exp\{\lambda_k(t) - b\})}{t^2}\, dt < +\infty \ (k = 1 \text{ or } 2), \tag{22}$$

then $M(\Lambda)$ is not dense in L^p_α.

Without loss of generality, we suppose (22) holds when $k = 1$. Let $\varphi(t)$ be an even function such that $\varphi(t) = \alpha(\exp\{\lambda_1(t) - b\})$ for $t \geqslant 0$ and let $u(z)$ be the Poisson integral of $\varphi(t)$, i.e.,

$$u(x + iy) = \frac{x}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(t)}{x^2 + (y - t)^2}\, dt. \tag{23}$$

Then $u(x + iy)$ is harmonic in the half-plane $\mathbb{C}_+$ and there exists an analytic function $g_1(z)$ on $\mathbb{C}_+$ satisfying

$$\operatorname{Re} g_1(z) = 4u(z) \geqslant (x - 1)(\lambda_1(|z|) - a) - \beta(x - 1),$$

where $z = x + iy, r = |z|, x > 1$. Let

$$g_0(z) = \frac{G_1(z)}{(1 + z)^N} \ \exp\{-g_1(z) - Nz - N\}, \tag{24}$$

where N is a large positive integer and $G_1(z)$ is defined by (10) for Λ_1. By (9) and (10), we have $g_0(\lambda) = 0$ for $\lambda \in \Lambda_1$, λ even and

$$|g_0(z)| \leqslant \frac{1}{1+|z|^2} \exp\{\beta(x-1)-x\}, \quad z \in \mathbb{C}_+. \tag{25}$$

Let

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0\left(\frac{1}{2}+iy\right) t^{-(\frac{1}{2}+iy)-1}\, dy. \tag{26}$$

Then $h_0(t)$ is continuous on $(0, +\infty)$. By Cauchy contour theorem,

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0(x+iy) t^{-(x+iy)-1}\, dy, \quad x > 0. \tag{27}$$

We obtain from (25) and the formula of Young transform (21) that

$$|h_0(t)| \leqslant \exp(-\alpha(t) - |\log t|) \tag{28}$$

and hence $h_0(t) \in L^q_{-\alpha}$. Moreover, by Mellin's transform formula,

$$g_0(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} h_0(t) t^z\, dt, \quad x > 0. \tag{29}$$

We extend the function $h_0(t)$ to an odd function by letting $h_0(t) = -h_0(-t)$ for $t < 0$. Therefore the bounded linear functional

$$T(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t) h(t)\, dt \quad (h \in L^p_\alpha) \tag{30}$$

satisfies $T(t^\lambda) = 0$ for $\lambda \in \Lambda$, and

$$\|T\| = \frac{2}{\sqrt{2\pi}} \left\{ \int_0^{+\infty} |h_0(t)|^q e^{q\alpha(t)}\, dt > 0 \right\}^{\frac{1}{q}}.$$

By the Hahn–Banach theorem, the space $M(\Lambda)$ is not dense in L^p_α.

(If (20) holds for $k = 2$, then we construct $g_0(z)$ by $G_2(z)$ for Λ_2 instead of $G_1(z)$ and then extend $h_0(t)$ to an even function by letting $h_0(t) = h_0(-t)$ for $t < 0$. The rest of the proof is similar.)

This completes the proof of theorem.

3. One conjecture

Inspired by Borichev [5], we have the following conjecture:

Conjecture 8. *Suppose that* $\alpha(t)$ *is an even nonnegative function satisfying* (1) *and* $\alpha(e^t)$ *is a convex function on* $\mathbb{R}$. *Let* $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ *be a sequence of increasing* *nonnegative integers. If $M(\Lambda)$ is not dense in $L^p_\alpha (1 \leqslant p < +\infty)$, then the closure of $M(\Lambda)$ consists of entire functions of exponential type zero $f(z)$ that is in L^p_α and has the form*

$$f(z) = \sum_{n=1}^{+\infty} a_n z^{\lambda_n}.$$

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References

[1] N. Ahiezer, On the weighted approximation of continuous functions by polynomials on the entire number axis, Uspekhi Mat. Nauk. 11 (1956) 3–43 (English translation in Amer. Math. Soc. Transl. Ser. 2 22 (1962) 95–137).
[2] S. Bernstein, Le problème des fonctions continues sur tout l’axe réel et l’une de ses applications, Bull. Soc. Math. France 52 (1924) 399–410.
[3] R.P. Boas Jr., Entire Functions, Academic Press, New York, 1954.
[4] A. Borichev, On weighted polynomial approximation with monotone weights, Proc. Amer. Math. Soc. (12) (2000) 3613–3619.
[5] A. Borichev, On the closure of polynomials in weighted space of functions on the real line, Indiana Univ. Math. J. 2 (2001) 829–845.
[6] P.B. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, New York, 1995.
[7] L. de Branges, The Bernstein problem, Proc. Amer. Math. Soc. (10)(1959) 825–832.
[8] G.T. Deng, Weighted polynomial approximation with gaps, Nagoya Math. J. (178)(2005).
[9] T. Hall, Sur l’approximation polynômiale des fonctions continues d’une variable réelle, Neuvième Congrrès des Mathémticiens Scandianaves (1938), Helsingfors, 1939, pp. 367–369.
[10] S. Izumi, T. Kawata, Quasi-analytic class and closure of $\{t^n\}$ in the interval $(-\infty, \infty)$, Tôhoku Math. J. 43 (1937) 267–273.
[11] P. Koosis, The Logarithmic Integral, vol. I, Cambridge University Press, Cambridge, 1988.
[12] A. Kroó, J. Szabados, On weighted approximation by lacunary polynomials and rational functions on the half-axis, East J. of Approx. 2 (3) (1996) 289–300.
[13] P. Malliavin, Sur quelques procédés d’extrapolation, Acta Math. 93 (1955) 179–255.
[14] S. Mergelyan, Weighted approximation by polynomials, Uspekhi Mat. Nauk 11 (1956) 107–152 (English translation in Amer. Math. Soc. Transl. Ser. 2 (10) (1958) 59–106).
[15] R. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.